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RESEARCH CONNECTION

Generalized Mittag-Leffler functions and fractional differential equations

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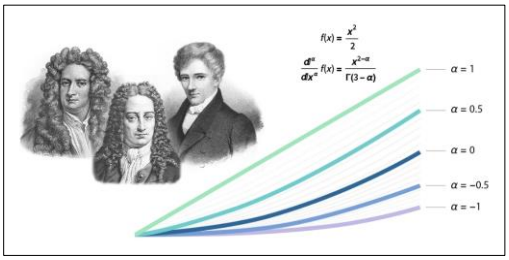

Why this research is important

As mentioned, fractional calculus is capable of modelling extremely complex phenomena and is thus becoming a more and more sought-after field of math for applications in science and engineering. As the phenomena get more complex, the corresponding models also tend to become more complex. Thus, the theoretical study of fractional integro-differential equations contributes towards creating a comprehensive body of work that creates a catalogue of analysis of these equations ranging from answering questions like: “Does this equation even have a solution?” “Is the solution unique, or are there multiple solutions?” “What happens when a slight modification (to be more technical, a perturbation) happens to the system we are given?” to demonstrate various useful and powerful techniques to tackle these massive equations.

How the research was conducted

Using various techniques such as theorems from Fixed Point Theory, Babenko’s Approach, and Banach’s Contractive Principle, we have analyzed some properties of fractional partial integro-differential equations with certain initial conditions. Furthermore, one recent topic we have been researching is making various

What you need to know

Ordinary calculus deals with the mathematics of change and the infinitely small. It is focused on two of the most important tools in mathematics: the derivative and the integral. These tools are useful in every field of science, finance, and many other areas. The application of these operators often lies in differential equations, which are equations that can capture some sort of changing relationship between several variables.

If we take a derivative once, we call it a first derivative. If we take it twice, we call it the second derivative. Similarly, you can get the third, fourth, fifth derivative and so on. We can also do multiple integrals. However, notice how we are restricted to only dealing with these whole number-order operators. What fractional calculus poses is an extension of regular calculus by allowing for fractional order derivatives and fractional order integrals. Suddenly, we are capable of asking what the half derivative of a function is or what the πth integral of a function might look like. By creating a continuum over which we can use these operators, fractional calculus has extended the capabilities of traditional calculus to allow for the modelling of much more complex systems and has been used in areas as different as quantum gravity theory to animal movement patterns. Our research focuses on analyzing various differential and integral equations with different conditions (called initial or boundary conditions) that use these fractional calculus operators.

generalizations to the Mittag-Leffler function, which is a widely used function in the study of fractional calculus. One such generalization is the matrix Mittag-Leffler function, which was used in the study conducted in our most recent publication. Since these generalized Mittag-Leffler functions are complex and there are no current means of consistently obtaining an exact solution, we developed a Python program to numerically approximate the values of these functions.

What the researchers found

Using various techniques, we have been able to find the conditions necessary for several fractional partial integro-differential equations to have a solution in the spaces we were working in. Furthermore, we showed that those solutions would be unique. Additionally, we have developed several extensions to the Mittag-Leffler functions—as well as corresponding code to numerically approximate them—that have proven to be useful in the analysis of various equations.

How this research can be used

There are various ways in which our research can be used. From a scientific and engineering perspective, it is possible that in their own research, they obtain an equation that is in the form of one of the equations we have studied. If that is the case, they can use our analysis to understand different properties of the equations. From a theoretical perspective, we have demonstrated multiple techniques that other researchers can learn and use in their research, and the generalizations of the Mittag-Leffler function also have many nice properties that researchers may find useful.

About the researchers

Joshua Beaudin graduated from Brandon University in 2022. Since then, he has continued the research he started with Dr. Li in 2020. BeaudinJ@brandonu.ca

Dr. Li is a professor in the Department of Mathematics and Computer Science at Brandon University. He has recently published the book *Towards Ulam Type Multi Stability Analysis*. LiC@brandonu.ca

Keywords

Fractional calculus, gamma function, fractional differential equations, Babenko's approach, Banach Contractive Principle, Fixed Point Theory, Python, Mittag-Leffler function

Publications based on this research

Beaudin, J., & Li, C. (2024). Application of a matrix Mittag-Leffler function to the fractional partial integro-differential equation in ℝn. *Journal of Mathematics and Computer Science, 33*(4), 420–430. https://dx.doi.org/10.22436/jmcs.033.04.08

Li, C., Saadati, R., Beaudin, J., & Hrytsenko, A. (2023). Remarks on a fractional nonlinear partial integro-differential equation via the new generalized multivariate Mittag-Leffler function. *Boundary Value Problems*, *96*. https://doi.org/10.1186/s13661-023-01783-6

Li, C., Beaudin, J., Rahmoune, A., & Remili, W. (2023). A matrix Mittag–Leffler function and the fractional nonlinear partial integro-differential equation in ℝn. *Fractal and Fractional, 7*(9), 651. https://doi.org/10.3390/fractalfract7090651

Li, C., Saadati, R., Beaudin, J., & Hrytsenko, A. (2023). On the uniqueness of the bounded solution for the fractional nonlinear partial integro-differential equation with approximations. *Mathematics, 11*(12), 2752. https://doi.org/10.3390/math11122752

Li, C., Saadati, R., Srivastava, R., & Beaudin, J. (2022). On the boundary value problem of nonlinear fractional integro-differential equations. *Mathematics, 10*(12), 1971. https://doi.org/10.3390/math10121971

Li, C., & Beaudin, J. (2021). On the nonlinear integro-differential equations. *Fractal and Fractional, 5*(3), 82. https://doi.org/10.3390/fractalfract5030082

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